TWMS Jour. Pure Appl. Math., V.3, N.1, 2012, pp.28-34

LINEAR LYAPUNOV FUNCTIONS FOR VOLTERRA QUADRATIC STOCHASTIC OPERATORS

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Abstract. We construct a class of linear Lyapunov functions for Volterra quadratic stochastic operator. Using these functions we improve known results about ω-limit set of trajectories of the Volterra quadratic operators.

Keywords: quadratic stochastic operator, Lyapunov function, trajectory, Volterra operators.

AMS Subject Classification: Primary 37N25, Secondary 92D10.

1. Introduction

The notion of quadratic stochastic operator (QSO) was first formulated by Bernshtein [1]. For more than 80 years this theory has been developed and many papers were published (see [1]-[10]). Several problems of physical and biological systems lead to necessity of study the asymptotic behavior of the trajectories of quadratic stochastic operators.

Let $E = \{1, 2, ..., m\}$. By the $(m-1)-$ simplex we mean the set

$$S^{m-1} = \{x = (x_1, ..., x_m) \in R^m : x_i \geq 0, \sum_{i=1}^{m} x_i = 1\}. \tag{1}$$

Each element $x \in S^{m-1}$ is a probability measure on E and so it may be looked upon as the state of a biological (physical and so on) system of m elements.

A quadratic stochastic operator $V : S^{m-1} \to S^{m-1}$ has the form

$$V : x'_k = \sum_{i,j=1}^{m} p_{ij,k} x_i x_j, \quad (k = 1, ..., m), \tag{2}$$

where $p_{ij,k}-$ coefficient of heredity and

$$p_{ij,k} = p_{ji,k} \geq 0, \ \sum_{k=1}^{m} p_{ij,k} = 1, \quad (i, j, k = 1, ..., m). \tag{3}$$

For a given $x^{(0)} \in S^{m-1}$, the trajectory $\{x^{(n)}\}, \ \ n = 0, 1, 2, ...$ of $x^{(0)}$ under the action of QSO (2) is defined by $x^{(n+1)} = V(x^{(n)})$, where $n = 0, 1, 2, ...$

One of the main problems in mathematical biology is to study the asymptotic behavior of the trajectories. In [2]-[4] this problem was solved for the Volterra QSO's by using the theories of the Lyapunov function and tournaments. These Lyapunov functions used in [2]-[4] were usually nonlinear.

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Manuscript received March 2011.

We refer to [6] for a detailed history, results and open problems related to quadratic stochastic operators. In particular, in [6] was stated an open problem: to find new classes of Lyapunov function for Volterra QSO.

In this paper we shall construct linear Lyapunov functions for Volterra QSO.

2. Definition

A Volterra QSO is defined by (2), (3) and the additional assumption

$$p_{ij,k}=0, \quad \text{if} \quad k\notin\{i,j\}, \ \forall i,j,k\in E. \tag{4}$$

The biological treatment of condition (4) is clear: the offspring repeats the genotype of one of its parents.

In [2] the general form of Volterra QSO

$$V: x=(x_1,...,x_m)\in S^{m-1} \ \rightarrow \ V(x)=x'=(x'_1,...,x'_m) \ \in S^{m-1}$$

is given

$$x'_k=x_k\left(1+\sum_{i=1}^m a_{ki}x_i\right), \quad k\in E, \tag{5}$$

where

$$a_{ki}=2p_{ik,k}-1 \ \ \text{for}\, i\neq k \ \text{and} \ \ a_{ii}=0, \ i\in E. \tag{6}$$

Moreover

$$a_{ki}=-a_{ik} \ \ \text{and} \ \ |a_{ki}|\leq 1.$$

Denote by $A=(a_{ij})_{i,j=1}^m$ the skew-symmetric matrix with entries (6).

Let $\{x^{(n)}\}_{n=1}^\infty$ be the trajectory of the point $x^0\in S^{m-1}$ under QSO (5). Denote by $\omega(x^0)$ the set of limit points of the trajectory. Since $\{x^{(n)}\}\subset S^{m-1}$ and S^{m-1} is compact, it follows that $\omega(x^0)\neq\emptyset$. Obviously, if $\omega(x^0)$ consists of a single point, then the trajectory converges, and $\omega(x^0)$ is a fixed point of (5). However, looking ahead, we remark that convergence of the trajectories is not the typical case for the dynamical systems (5). Therefore, it is of particular interest to obtain an upper bound for $\omega(x^0)$, i.e., to determine a sufficiently "small" set containing $\omega(x^0)$.

Denote

$$\operatorname{int} S^{m-1}=\{x\in S^{m-1}:\prod_{i=1}^m x_i>0\}, \quad \partial S^{m-1}=S^{m-1}\setminus \operatorname{int} S^{m-1}.$$

Definition 2.1. *A continuous function $\varphi: S^{m-1}\rightarrow R$ is called a Lyapunov function for the dynamical system (5) if the limit $\lim\limits_{n\to\infty}\varphi(x^{(n)})$ exists for any initial point x^0.*

Obviously, if $\lim\limits_{n\to\infty}\varphi(x^{(n)})=c$, then $\omega(x^0)\subset\varphi^{-1}(c)$. Consequently, for an upper estimate of $\omega(x^0)$ we should construct the set of Lyapunov functions that is as large as possible.

The following results are known:

Theorem 2.1. [2],[4] *For the Volterra QSO (5) the following assertions hold*

1) For the dynamical system (5) there exists a Lyapunov function of the form $\varphi_p(x)=x_1^{p_1}...x_m^{p_m}$, where $p_i\geq 0$, $\sum\limits_{i=1}^m p_i=1$ and $x=(x_1,...,x_m)\in \operatorname{int} S^{m-1}$.

2) If there is $r\in\{1,...,m\}$ such that $a_{ij}<0$ (see (4)) for all $i\in\{1,...,r\}$, $j\in\{r+1,...,m\}$ then $\varphi(x)=\sum\limits_{i=r+1}^m x_i$, $x\in S^{m-1}$ is a Lyapunov function for QSO (5).

3) There are Lyapunov functions of the form

$$\varphi(x) = \frac{x_i}{x_j},\ i \neq j,\ x \in \operatorname{int} S^{m-1}.$$

3. Linear Lyapunov functions

The following theorem gives a condition under which a given linear function is a Lyapunov function.

Theorem 3.1. *For the Volterra operator (5), the function $\varphi : S^{m-1} \to R$ defined by*

$$\varphi_c(x) = \sum_{k=1}^{m} c_k x_k \tag{7}$$

is a Lyapunov function if $c = (c_1, ..., c_m)$ satisfies $c_k a_{ki} \leq 0$ for all $i, k \in E$.

Proof. Suppose that $c_k a_{ki} \leq 0$ for all $i, k \in E$, then we have

$$\varphi_c(V(x)) = \sum_{k=1}^{m} c_k x'_k = \sum_{k=1}^{m} c_k x_k \left(1 + \sum_{i=1}^{m} a_{ki} x_i\right) =$$

$$= \sum_{k=1}^{m} c_k x_k + \sum_{k=1}^{m} c_k x_k \sum_{i=1}^{m} a_{ki} x_i = \varphi_c(x) + \sum_{k=1}^{m} x_k \sum_{i=1}^{m} c_k a_{ki} x_i \leq \varphi_c(x).$$

Thus, for any n we have $\varphi_c(x^{(n)}) \leq \varphi_c(x^{(n-1)})$ and $\underline{c} \leq \varphi_c(x^{(n)}) \leq \overline{c}$, with $\underline{c} = \min_k c_k, \overline{c} = \max_k c_k$. Consequently, the sequence $\{\varphi_c(x^{(n)})\}_{n=0}^{\infty}$ is convergent. Therefore, $\varphi_c(x) = \sum_{k=1}^{m} c_k x_k$ is a Lyapunov function for the operator (5). This completed the proof. $\square$

Corollary 3.1. *The function defined by*

$$\phi(x) = \prod_{k=1}^{m} \left(\sum_{i=1}^{m} c_{ki} x_i\right)^{p_k}$$

is a Lyapunov function for Volterra operator (5) for any $p_k \in R^+$, if $c^{(q)} = (c_{q1}, c_{q2}, ..., c_{qm})$ satisfies $c_{qk} a_{ki} \leq 0$ for all $i, q, k \in E$.

Remark 3.1. *The set of Volterra quadratic operators and vectors c, which satisfying condition of theorem 3.1 is non-empty. On the S^2 we see the following QSO*

$$V : \begin{cases} x'_1 = x_1(1 + x_2 + x_3), \\ x'_2 = x_2(1 - x_1 + x_3), \\ x'_3 = x_3(1 - x_1 - x_2), \end{cases} \tag{7'}$$

and as a vector c we get $c = (-l, 0, l)$, $\ l \in R^+$. It easy one can see that $c_k a_{ki} \leq 0$ for all $k, i = 1, 2, 3$.

Denote by $c_\uparrow$ the vector $c_\uparrow = (c_1, \ldots, c_m)$, where $c_1 \leq c_2 \leq ... \leq c_m$. Also we denote by $c_\downarrow$ the vector $c \in R^m$, where $c_1 \geq c_2 \geq ... \geq c_m$.

Theorem 3.2. *The function*

$$\varphi_{c_\uparrow}(x)=\sum_{k=1}^{m} c_k x_k \tag{8}$$

is a Lyapunov function for the Volterra operator (5), for any vector $c_\uparrow$, if $a_{ki}\geq 0$ for all $k\leq i$.

Proof. Suppose that $a_{ki}\geq 0$ for all $k\leq i$, then for vector $c_\uparrow$ we have using (6)

$$\begin{aligned}\varphi_{c_\uparrow}(V(x))=\sum_{k=1}^{m} c_k x'_k=\sum_{k=1}^{m} c_k x_k\left(1+\sum_{i=1}^{m} a_{ki}x_i\right)=\sum_{k=1}^{m} c_k x_k+\sum_{k=1}^{m} c_k x_k\sum_{i=1}^{m} a_{ki}x_i=\\ =\varphi_{c_\uparrow}(x)+c_1x_1\sum_{i=1}^{m} a_{ki}x_i+c_2x_2\sum_{i=1}^{m} a_{ki}x_i+...+c_mx_m\sum_{i=1}^{m} a_{ki}x_i=\varphi_{c_\uparrow}(x)+\\ +\sum_{1\leq k<i\leq m}(c_ka_{ki}+c_ia_{ik})x_kx_i=\varphi_{c_\uparrow}(x)+\sum_{1\leq k<i\leq m}(c_ka_{ki}-c_ia_{ki})x_kx_i=\varphi_{c_\uparrow}(x)+\\ +\sum_{1\leq k<i\leq m}(c_k-c_i)a_{ki}x_kx_i.\end{aligned} \tag{9}$$

Therefore

$$\varphi_{c_\uparrow}(V(x))\leq\varphi_{c_\uparrow}(x)+\sum_{1\leq k<i\leq m}(c_k-c_i)a_{ki}x_kx_i\leq\varphi_{c_\uparrow}(x).$$

□

Corollary 3.2. *The function*

$$\varphi_{c_\downarrow}(x)=\sum_{k=1}^{m} c_k x_k$$

is a Lyapunov function for the Volterra operator (5), for any vector $c_\downarrow$, if $a_{ki}\leq 0$ for all $k\leq i$.

Remark 3.2. *The set of Volterra quadratic operators which satisfy condition of theorem 3.2 is non-empty. For example:Volterra QSO corresponding to the following skew-symmetric matrix*

$$A=\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 & ... & 1 & 1\\ -1 & 0 & 1 & 1 & 1 & 1 & ... & 1 & 1\\ -1 & -1 & 0 & 1 & 1 & 1 & ... & 1 & 1\\ -1 & -1 & -1 & 0 & 1 & 1 & ... & 1 & 1\\ -1 & -1 & -1 & -1 & 0 & 1 & ... & 1 & 1\\ -1 & -1 & -1 & -1 & -1 & 0 & ... & 1 & 1\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & ... & \vdots & \vdots\\ -1 & -1 & -1 & -1 & -1 & -1 & ... & 0 & 1\\ -1 & -1 & -1 & -1 & -1 & -1 & ... & -1 & 0 \end{pmatrix} \tag{10}$$

satisfies the condition.

4. The $\omega-$limit set

The problem of describing the $\omega-$limit set of a trajectory is of great importance in the theory of dynamical systems. The following theorem completely describes the behavior of the trajectories of Volterra operators, which satisfies condition of theorem 3.2.

We denote by $Fix(V)$ the set of fixed points operator (5).

Theorem 4.1. *If $a_{ki} > 0$ for all $k \leq i$ and $c_{k_0} < c_{k_0+1}$ satisfied for some $k_0 \in \{1, 2, ..., m-1\}$, and vector $c_\uparrow$, then for any initial point $x^0 \in S^{m-1}$, $x^0 \notin Fix(V)$ we have $\omega(x^0) \subset \partial S^{m-1}$.*

Proof. Using the equality $x_1 + x_2 + ... + x_m = 1$ from (8) we get

$$\varphi_{c_\uparrow}(V(x)) = \sum_{k=1}^{m} c_k x'_k = \sum_{k=2}^{m} (c_k - c_1) x_k + c_1. \tag{11}$$

For any vector $c_\uparrow$ we have $c_1 \leq \varphi_{c_\uparrow}(V(x))$, $\forall x \in S^{m-1}$ and from theorem 3.2 we have $\varphi_{c_\uparrow}(x^{(n+1)}) \leq \varphi_{c_\uparrow}(x^{(n)})$, $n = 0, 1, 2, ...$, therefore there exits

$$\lim_{n\to\infty} \varphi_{c_\uparrow}(x^{(n)}) = q \geq c_1.$$

If $q = c_1$ then it is clear that the function (11) is a convex and decreasing. We have

$$\min_{x \in S^{m-1}} \varphi_{c_\uparrow}(x) = \varphi_{c_\uparrow}(e_1) = c_1,$$

though the minimum obtain only at the point $e_1 = (1, 0, ..., 0)$.

We suppose that $q > c_1$, that equivalent to $\lim\limits_{n\to\infty} \varphi_{c_\uparrow}(x^{(n)}) - c_1 > 0$. Then using (5), (9) and (11) we get

$$1 = \lim_{n\to\infty} \frac{\varphi_{c_\uparrow}(x_k^{(n+1)}) - c_1}{\varphi_{c_\uparrow}(x_k^{(n)}) - c_1} = \lim_{n\to\infty} \frac{\sum\limits_{k=1}^{m} c_k x_k^{(n+1)} - c_1}{\sum\limits_{k=1}^{m} c_k x_k^{(n)} - c_1} =$$

$$= \lim_{n\to\infty} \frac{\sum\limits_{k=1}^{m} c_k x_k^{(n)} \left(1 + \sum\limits_{i=1}^{m} a_{ki} x_i^{(n)}\right) - c_1}{\sum\limits_{k=1}^{m} c_k x^{(n)} - c_1} = 1 + \lim_{n\to\infty} \frac{\sum\limits_{k=1}^{m} c_k x_k^{(n)} \sum\limits_{i=1}^{m} a_{ki} x_i^{(n)}}{\sum\limits_{k=1}^{m} c_k x_k^{(n)} - c_1}.$$

$$\lim_{n\to\infty} \frac{\sum\limits_{k=1}^{m} c_k x_k^{(n)} \sum\limits_{i=1}^{m} a_{ki} x_i^{(n)}}{\sum\limits_{k=1}^{m} c_k x_k^{(n)} - c_1} = \frac{\sum\limits_{k=1}^{m} c_k x_k^{(*)} \sum\limits_{i=1}^{m} a_{ki} x_i^{(*)}}{\sum\limits_{k=1}^{m} c_k x_k^{(*)} - c_1} = 0. \tag{12}$$

$$\sum_{k=1}^{m} c_k x_k^{(*)} \sum_{i=1}^{m} a_{ki} x_i^{(*)} = \sum_{1 \leq k < i \leq m} (c_k - c_i) a_{ki} x_k^* x_i^*. \tag{13}$$

Since $a_{ki} > 0$ for all $k \leq i$ and $c_{k_0} < c_{k_0+1}$ satisfies for some $k_0 \in \{1, 2, ..., m-1\}$, for any vector $c_\uparrow$ from (13) we get $(c_{k_0+1} - c_{k_0}) a_{ki} x_{k_0+1}^* x_{k_0}^* > 0$. This contradictory statement for (11). Therefore $x^* \in \partial S^{m-1}$.

The proof of theorem 4.1 is complete. □

Corollary 4.1. *If $a_{ki} > 0$ for all $k < i$ and $c_k < c_{k+1}$ for all $k \in \{1, 2, ..., m-1\}$ and vector $c_\uparrow$, then for initial point $x^0 \in S^{m-1}$, $x_1^0 > 0, x^0 \notin Fix(V)$ we have $\omega(x^0) = \{e_1\}$, i.e. contains of a single point.*

Proof. From (12), (13) we get

$$x_k^* x_{k+1}^* = 0, \quad k \in \{1, 2, ..., m-1\}.$$

In other words

$$x_1^* > 0, \quad x_2^* = 0, \quad x_3^* \geq 0, \quad x_4^* = 0, ..., x_{m-1}^* \geq 0, \quad x_m^* = 0.$$

From (5) we have

$$x_1^{(n+1)} = x_1^{(n)} \left(1 + \sum_{i=1}^{m} a_{1i} x_i^{(n)} \right).$$

Since $\left(1 + \sum\limits_{i=1}^{m} a_{1i} x_i^{(n)} \right) \geq 1$ for any n, the sequence $\{x_1^{(n)}\}$ is non decreasing and $0 < x_1^{(n)} \leq 1$. Therefore there exists $\lim\limits_{n\to\infty} x_1^{(n)} = 1$. Therefore $\lim\limits_{n\to\infty} x^{(n)} = e_1$.

□

Remark 4.1. *The QSO V satisfies the ergodic theorem (see* [11]*) if the limit*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} V^{(k)}(x)$$

exits for any $x \in S^{m-1}$. On the basis of numerical calculation Ulam conjectured ([11]*) the ergodic theorem holds for any QSO. In* [13] *it was proven that this conjecture is false in general. From theorem 4.1 follows that the ergodic theorem holds for any QSO determined by conditions theorem 3.2.*

Remark 4.2. *Note that the form of the Lyapunov function in theorem 3.2 more general than the Lyapunov function in 2) of theorem 2.1.*

Remark 4.3. *By theorem 4.1 we see that our Lyapunov functions (8) are applicable to wider class of Volterra operators, than the Lyapunov function in 2) of theorem 2.1.*

Remark 4.4. *We note that for the Volterra quadratic stochastic operators only Lyapunov functions mentioned in theorem 2.1 and the linear Lyapunov functions (described in this paper) are known. We have not an example of another type of Lyapunov function for the Volterra operators. The description of the set of all Lyapunov functions is very difficult problem.*

5. Acknowledgements

The author thanks to Professor U.A. Rozikov for useful discussions and the referee, whose valuable suggestions improved the content of this paper considerably.

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